Superstatistical random-matrix-theory approach to transition intensities in mixed systems

A. Y. Abul-Magd
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt
(Received 12 January 2006; published 19 May 2006)

We study the fluctuation properties of transition intensities applying a recently proposed generalization of the random matrix theory, which is based on Beck and Cohen's superstatistics. We obtain an analytic expression for the distribution of the reduced transition probabilities that applies to systems undergoing a transition out of chaos. The obtained distribution fits the results of a previous nuclear shell model calculations for some electromagnetic transitions that deviate from the Porter–Thomas distribution. It agrees with the experimental reduced transition probabilities for the ^{26}Al nucleus better than the commonly used χ^2 distribution.

DOI: 10.1103/PhysRevE.73.056119 PACS number(s): 02.50.−r, 05.40.−a, 05.45.Mt, 03.65.−w

I. INTRODUCTION

Random matrix theory (RMT) is believed to describe quantal systems whose classical counterpart has a chaotic dynamics [1–3]. In RMT the matrix elements of the Hamiltonian in some basis are replaced with random numbers. The theory is based on two main assumptions: (i) The matrix elements are independent identically distributed random variables, and (ii) their distribution is invariant under unitary transformations. These lead to a Gaussian probability density distribution for the matrix elements, $P(H)\propto\exp[-\eta\times\mathrm{Tr}(H^{\dagger}H)]$. With these assumptions, RMT presents a satisfactory description for numerous chaotic systems. Agreement with RMT is now considered to be a signature of chaos in the quantum system. For time-reversal-invariant systems, the appropriate form of random matrix theory is the Gaussian orthogonal ensemble (GOE); that is the form which will mainly be considered in this paper.

For most systems, however, the phase space is partitioned into regular and chaotic domains. These systems are known as mixed systems. Attempts to generalize RMT to describe such mixed systems are numerous; for a review please see [3]. Most of these attempts are based on constructing ensembles of random matrices whose elements are independent but not identically distributed. Thus, the resulting expressions are not invariant under base transformation. The first work in this direction is due to Rosenzweig and Porter [4]. They model the Hamiltonian of the mixed system by a superposition of a diagonal matrix of random elements having the same variance and a matrix drawn from a GOE. Therefore, the variances of the diagonal elements of the total Hamiltonian are different from those of the off-diagonal ones, unlike the GOE Hamiltonian in which the variances of diagonal elements are twice those of the off-diagonal ones. Hussein and Pato [5] used the maximum entropy principle to construct "deformed" random-matrix ensembles by imposing different constraints for the diagonal and off-diagonal elements. This approach has been successfully applied to the case of metal-insulator transition [6]. A recent review of the deformed ensemble is given in [7]. Ensembles of band random matrices, whose entries are equal to zero outside a band of limited width along the principal diagonal, have often been used to model mixed systems [2,8,9].

The past decade has witnessed a considerable interest devoted to the possible generalization of statistical mechanics. Much work in this direction followed the Tsallis seminal paper [10]. Tsallis introduced a nonextensive entropy, which depends on a positive parameter q known as the entropic index. The standard Shannon entropy is recovered for $q=1$. Applications of the Tsallis formalism covered a wide class of phenomena; for a review please see, e.g., [11]. Recently, the formalism has been applied to include systems with mixed regular-chaotic dynamics in the framework of RMT [12–17]. However, the constraints of normalization and existence of an expectation value for $\mathrm{Tr}(H^{\dagger}H)$ set up an upper limit for the entropic index q beyond which the involved integrals diverge. This restricts the validity of the nonextensive RMT to a limited range near the chaotic phase [16,17].

Another extension of statistical mechanics is provided by the formalism of superstatistics (statistics of a statistics), recently proposed by Beck and Cohen [18]. Superstatistics arises as weighted averages of ordinary statistics (the Boltzmann factor) due to fluctuations of one or more intensive parameter (e.g., the inverse temperature). It includes Tsallis' nonextensive statistics, for $q\geq 1$, as a special case in which the inverse temperature has a χ^2-distributions. This formalism has been applied to model a mixed system within the framework of RMT in Refs. [19,20]. The joint matrix element distribution was represented as an average over $\exp[-\eta\,\mathrm{Tr}(H^{\dagger}H)]$ with respect to the parameter η. The different choices of parameter distribution, which had been studied in Beck and Cohen's paper [18], were considered in [19]. The parameter distribution has also been estimated [20] by applying the principle of maximum entropy, as done by Sattin [21]. Explicit analytical results were obtained for the level density and the nearest neighbor-spacing distributions.

Matrix elements of transition operator probe the system's wave functions so that their statistical fluctuations provide additional information. In chaotic systems, the reduced transition probabilities follow the Porter-Thomas distribution [22]. This is a χ^2-distribution of one degree of freedom. As the system becomes more regular, the transition probabilities deviate from the Porte-Thomas distribution. To account for these deviations, Alhassid and Novoselsky [23] suggested that the transition widths in the mixed system may be analyzed in terms of a χ^2-distribution of a lower degree of freedom. The latter distribution does not fit well the empirical distributions but consists with the observed number of weak transitions as compared with the Porter-Thomas distribution (see, e.g., [24–26]). The distributions of experimental reduced transition probabilities B in ^{26}Al [28] and ^{30}P [29] expressed as functions of the $\log B$ have peaks at $\log B < 0$ while all the χ^2 distributions are peaked at $\log B = 0$. We show in the present paper that the superstatistical RMT provides us with a more suitable generalization of the Porter-Thomas distribution. In Sec. II we briefly review the concept of superstatistics and the necessary generalization required to express the characteristics of the spectrum of a mixed system into an ensemble of chaotic spectra with different local mean level density. The evolution of the reduced transition-intensity distribution during the stochastic transition induced by increasing the local-density fluctuations is considered in Sec. III. Section IV demonstrates the quality of fit achieved by the obtained transition-intensity distribution by comparing its prediction with the results of a shell-model calculation by Hamoudi *et al.* [26]. The conclusion of this work is formulated in Sec. V.

II. SUPERSTATISTICAL RMT

To start with, we briefly review the superstatistics concept as introduced by Beck and Cohen [18]. Consider a nonequilibrium system with spatiotemporal fluctuations of the inverse temperature β. Locally, i.e., in spatial regions (cells) where β is approximately constant, the system may be described by a canonical ensemble in which the distribution function is given by the Boltzmann factor $e^{-\beta E}$, where E is an effective energy in each cell. In the long-term run, the system is described by an average over the fluctuating β. The system is thus characterized by a convolution of two statistics, and hence the name "superstatistics." One statistics is given by the Boltzmann factor and the other one by the probability distribution $f(\beta)$ of β in the various cells. One obtains Tsallis' statistics when β has a χ^2 distribution, but this is not the only possible choice. Beck and Cohen give several possible examples of functions which are possible candidates for $f(\beta)$. Sattin [21] suggested that, lacking any further information, the most probable realization of $f(\beta)$ will be the one that maximizes the Shannon entropy. Namely this version of superstatistics formalism will now be applied to RMT.

A. Joint distribution of the matrix-elements

Gaussian random-matrix ensembles have several common features with the canonical ensembles. In RMT, the square of a matrix element plays the role of energy of a molecule in a gas. When the matrix elements are statistically identical, one expects them to become distributed as the Boltzmann's. One obtains a Gaussian probability density distribution of the matrix elements

$$P_G(H) \propto \exp[-\eta \operatorname{Tr}(H^\dagger H)] \tag{1}$$

by extremizing the Shannon entropy [1] subjected to the constraints of normalization and existence of the expectation value of $\operatorname{Tr}(H^\dagger H)$. Here Tr means trace and $H^\dagger$ is the Hermitian conjugate of H. The quantity $\operatorname{Tr}(H^\dagger H)$ plays the role of the effective energy of the system, while the role of the inverse temperature β is played by η, being twice the inverse of the matrix-element variance.

Our main assumption is that Beck and Cohen's superstatistics provides a suitable description for systems with mixed regular-chaotic dynamics. We consider the spectrum of a mixed system as made up of many smaller cells that are temporarily in a chaotic phase. Each cell is large enough to obey the statistical requirements of RMT but has a different distribution parameter η associated with it, according to a probability density $\tilde{f}(\eta)$. Consequently, the superstatistical random-matrix ensemble that describes the mixed system is a mixture of Gaussian ensembles. Its matrix-element joint probability density distributions obtained by integrating distributions of the form in Eq. (1) over all positive values of η with a statistical weight $\tilde{f}(\eta)$,

$$P(H) = \int_0^\infty \tilde{f}(\eta) \frac{\exp[-\eta \operatorname{Tr}(H^\dagger H)]}{Z(\eta)} d\eta, \tag{2}$$

where $Z(\eta) = \int \exp[-\eta \operatorname{Tr}(H^\dagger H)] d\eta$. Here we use the "B type superstatistics" [18]. The distribution in Eq. (2) is isotropic in the matrix-element space. Relations analogous to Eq. (1) can also be written for the joint distribution of eigenvalues as well as any other statistic that is obtained from it by integration over some of the eigenvalues, such as the nearest-neighbor-spacing distribution and the level number variance. The distribution $\tilde{f}(\eta)$ has to be normalizable, to have at least a finite first moment and reduces a delta function as the system becomes fully chaotic.

An analogous ensemble made of a superposition of random-matrix ensembles has recently been considered by Muttalib and Klauber [27]. These authors have been seeking generalizations of Gaussian random-matrix ensembles, with the probability distributions $P(H)$ that are functions of the single variable $\operatorname{Tr}(H^\dagger H)$ like the distribution (2) that follows here from the concept of superstatistics. However, they work not directly with the distributions $P(H)$ themselves, but with the associated characteristic functions defined as the Fourier transforms

$$C(T) = \int \exp[i \operatorname{Tr}(T^\dagger H)] P(H) dH. \tag{3}$$

They prove, among other things, that if $C(T)$ is a function of $\operatorname{Tr}(T^\dagger T)$ only, then the most general $C(T)$, valid for random-matrix ensembles of arbitrarily large dimension can be represented as

$$C(T) = \int f(b) \exp[-b \operatorname{Tr}(T^\dagger T)] db. \tag{4}$$

The inverse Fourier transformation of $C(T)$ then leads to an expression for $P(H)$ similar to the one in Eq. (2). The equivalence of the approach to RMT initiated by Muttalib and Klauber with the superstatistical approach used in [19,20] and advocated here is due to the fact that the Fourier transformation is accomplished by a linear operator. We consider their result as a justification of using Eq. (2) for ensembles of matrices of dimensions $N \to \infty$.

B. Marginal distribution for a single matrix-element

Unlike the Gaussian random-matrix ensembles, the superstatistical ensemble has correlated matrix elements. This can clearly be seen by the fact that the joint distribution function defined by Eq. (2) does not factorize into a product of distributions of the individual matrix elements. However, it is not difficult to obtain a marginal distribution for each of the individual matrix elements.

We shall confine our consideration for the GOE; the generalization to the other symmetry universalities is straightforward. In this case,

$$Tr(H^{\dagger}H) = Tr(H^2) = \sum_k H_{kk}^2 + 2\sum_{k<l} H_{kl}^2. \tag{5}$$

Integrating the joint distribution $P(H)$ over all the matrix elements except one, say H_{if}, we obtain

$$p_{if}(H_{if}) = \int_0^{\infty} \tilde{f}(\eta)\sqrt{2\eta/\pi}\exp[-2\eta H_{if}^2]d\eta. \tag{6}$$

The parameter η essentially defines the energy scale of the individual ensembles, whose superposition compose the superstatistical ensemble. We, therefore, assume that the distribution (6) will hold for the superstatistical distribution of any physical quantity having the dimension of energy, which is represented as a Gaussian random variable in the case of GOE.

We note that the distribution function of superstatistical ensemble depends on the matrix elements through $\mathrm{Tr}(H^2)$ which is base invariant. In other words, the function $P(H)$ is invariant under rotation in the space of the matrix elements. Therefore, the distributions p_{if} have the same form for all off-diagonal matrix elements of H.

C. Parameter distribution

Following Sattin [21], we use the principle of maximum entropy to evaluate $\tilde{f}(\eta)$. Lacking a detailed information about the mechanism causing the deviation from the prediction of RMT, the most probable realization of $\tilde{f}(\eta)$ will be the one that extremizes the Shannon entropy

$$S = -\int_0^{\infty} \tilde{f}(\eta)\ln\tilde{f}(\eta)d\eta \tag{7}$$

with the following constraints:

Constraint 1. The major parameter of RMT is η defined in Eq. (1). Superstatistics was introduced in Eq. (2) by allowing η to fluctuate around a fixed mean value $\langle\eta\rangle$. This implies the existence of this mean value

$$\langle\eta\rangle = \int_0^{\infty} \tilde{f}(\eta)\eta d\eta. \tag{8}$$

Constraint 2. The fluctuation properties are usually defined for the unfolded spectra, which have a unit mean level spacing. The mean level density is proportional to the inverse square root of η. We thus require the existence of the integral

$$\int_0^{\infty} \tilde{f}(\eta)\eta^{-1/2}d\eta = 1. \tag{9}$$

Therefore, the most probable $\tilde{f}(\eta)$ extremizes the functional

$$F = -\int_0^{\infty} \tilde{f}(\eta)\ln\tilde{f}(\eta)d\eta - \lambda_1\int_0^{\infty} \tilde{f}(\eta)\eta d\eta - \lambda_2\int_0^{\infty} \tilde{f}(\eta)\eta^{-1/2}d\eta, \tag{10}$$

where λ_1 and λ_2 are Lagrange multipliers. As a result, we obtain

$$\tilde{f}(\eta) = C\exp\left[-\alpha\left(\frac{\eta}{\eta_0} + 2\left(\frac{\eta_0}{\eta}\right)^{1/2}\right)\right] \tag{11}$$

where α and η_0 are parameters, which can be expressed in terms of the Lagrange multipliers λ_1 and λ_2, and C is a normalization constant. The latter is given by

$$C = \frac{\alpha\sqrt{\pi}}{\eta_0 G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)}. \tag{12}$$

Here $G_{03}^{30}(x|b_1,b_2,b_2)$ is a Meijer's G-function [30,31]; see also the Appendix of Ref. [20]. The parameter distribution $\tilde{f}(\eta)$ is peaked at η_0 and tends to a delta function as $\alpha \to \infty$. The value of η_0 will be fixed in the next section while the parameter α will be considered as the tuning parameter for the stochastic transition.

A parameter distribution analogous to $\tilde{f}(\eta)$ is obtained in [20], where the variable η is replaced by the local mean level density. This distribution is used in [20] to derive expressions for the level density distribution, the nearest-neighbor spacing distribution, and the two-level correlation function for spectra of superstatistical ensembles. The expression obtained for the level-density distribution has a finite value at the center of the spectrum for arbitrarily large ensemble dimension N, and thus satisfies a necessary condition on $\tilde{f}(\eta)$, which is required by Muttalib and Klauber [27].

III. TRANSITION-INTENSITY DISTRIBUTION

The probability B_{if} of a transition from the initial configuration $|i\rangle$ to the final configuration $|f\rangle$ is given by

$$B_{if} = |W_{if}|^2, \tag{13}$$

where

$$W_{if} = \langle f|O|i\rangle \tag{14}$$

is the square of the transition operator O in a special basis. In a chaotic system, the eigenstates $|i\rangle$ and $|f\rangle$ are believed to be very complicated. If the operator O conserves time reversibility, the matrix elements W_{if} are real. For a chaotic system, it is reasonable to assume that W_{if} are identically distributed Gaussian random variable. This entails that the transition intensities can be represented by a random variable that takes the values

$$y_{if}=\frac{B_{if}}{\langle B_{if}\rangle} \tag{15}$$

where $\langle B_{ij}\rangle$ is a suitably defined local average value [24], and has a Porter-Thomas distribution

$$P_{\mathrm{PT}}(y)=\sqrt{\frac{\eta}{\pi y}}e^{-\eta y}. \tag{16}$$

The parameter η is defined by the requirement that $\langle y\rangle=1$, and is equal to 1/2. A more elaborate derivation of the Porter-Thomas distribution is given by Barbosa *et al.* [25].

We now derive the superstatistical generalization of the Porter-Thomas distribution. For this purpose, we assume that the matrix elements W_{if} are distributed according to Eq. (6). The parameter η in Eq. (16) is no more considered as a constant but allowed to fluctuate according to the distribution $\tilde{f}(\eta)$. The superstatistical transition intensity distribution is then given by

$$P_{\mathrm{Superstatistical}}(y)=\int_0^\infty \tilde{f}(\eta)\sqrt{\frac{\eta}{\pi y}}e^{-\eta y}d\eta \tag{17}$$

Substituting Eq. (11) for $\tilde{f}(\eta)$ and integrating over η, we obtain

$$P_{\mathrm{Superstatistical}}(y)=\frac{\alpha}{\sqrt{\pi y}}\frac{G_{03}^{30}\left(\alpha^2(\alpha+\eta_0 y)|0,\frac{1}{2},\frac{3}{2}\right)}{\eta_0(y+\alpha/\eta_0)^{3/2}G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)} \tag{18}$$

The parameter η_0 is determined from the requirement that $\langle y\rangle=1$, which yields

$$\eta_0=\frac{\alpha}{2}\frac{G_{03}^{30}\left(\alpha^3|0,0,\frac{1}{2}\right)}{G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)}. \tag{19}$$

Replacing Meijer's G-function by its large-argument asymptotic expression

$$G_{0,3}^{3,0}(z|b_1,b_2,b_3)\sim\frac{2\pi}{\sqrt{3}}z^{(b_1+b_2+b_3-1)/3}\exp(-3z^{1/3}). \tag{20}$$

one can easily show that $P_{\mathrm{Superstatistical}}(y)$ is reduced to the Porter-Thomas distribution as the parameter $\alpha\to\infty$.

Several independent results with different models, have suggested that transition strengths in a chaotic system follow a χ^2 distribution

$$P_{\chi^2}(y,\nu)=\frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}y^{\nu/2-1}e^{-\nu y}, \tag{21}$$

with $\nu=1$ (porter-Thomas) degrees of freedom, the transition strengths in a less chaotic system a χ^2 distribution with a number of degrees of freedom less than one. Alhassid and Levine [32] introduced this distribution using maximum-entropy arguments. Several studies of electromagnetic transition intensities in nuclei have also been performed [23–26,28,29]; each of these has suggested that the χ^2 distribution with $\nu<1$ is appropriate for relatively regular systems.

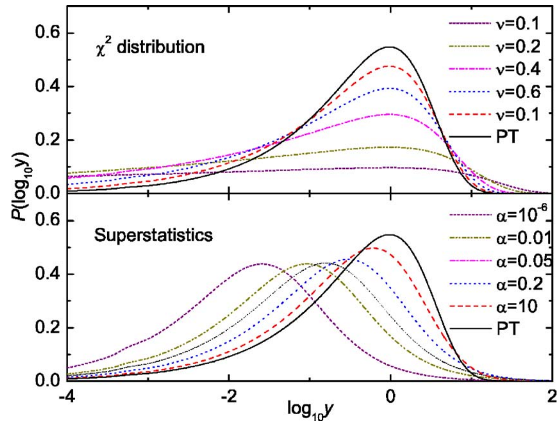

FIG. 1. (Color online) Evolution of the superstatistical distribution $P_{\mathrm{Superstatistical}}(\log_{10}y)$ and the χ^2 distirbutions $P_{\chi^2}(\log_{10}y,\nu)$ during the transition from chaotic to regular dynamics. The solid curves, labeled as PT, refer to the Porter-Thomas distribution.

Experimental data for transition intensities range over several orders of magnitude. It is often more convenient to consider a logarithmic variable as the argument. The probability density function in terms of $\log_a y$ is

$$F(\log_a y)=y\ln aF(y). \tag{22}$$

We compare in Fig. 1 the evolution of $P_{\mathrm{Superstatistical}}(\log_{10}y)$ during the transition from chaotic to regular dynamics by varying the tuning parameter α from ∞ (GOE) to 10^{-6} (almost regular) with a corresponding evolution of $P_{\chi^2}(\log_{10}y,\nu)$ where ν varies between 1 and 0.1. Of special interest is the fact that the maximum of $P_{\chi^2}(\log_{10}y,\nu)$ occurs at $\log_{10}y=0$ for any value of ν. This property does not hold for $P_{\mathrm{Superstatistical}}(\log_{10}y)$. The peak of superstatistical distribution for less chaotic systems occurs at $\log_{10}y<0$ and moves towards lower values as the parameter α decreases. We show in the next section that this is indeed the behavior of physical systems.

IV. DATA ANALYSIS

The purpose of this section is to show that the proposed superstatistical distribution succeeds in the situations where the χ^2 distribution fails. We show this by using results from two works, which examine the effect of a transition from chaos to integrability on gamma-ray reduced transition probabilities.

The first work is done by Hamoudi, Nazmitdinov and Alhassid [26]. They calculated the electric quadrupole (E2) and magnetic dipole (M1) transition intensities among the isospin $T=0,1$ states of nuclei with mass number 60. They applied the interacting shell model with realistic interaction for pf shell nuclei with a ^{56}Ni core. It is found that the B(E2) transitions are well described by a GOE (Porter-Thomas distribution). However, the statistics for the B(M1) transitions is sensitive to T_z. The M2 transition operator consists of an isoscalar and isovector components. The $T_z=1$ nuclei, in which both components contribute, exhibit a Porter-Thomas distribution. In the meanwhile, a significant deviation from

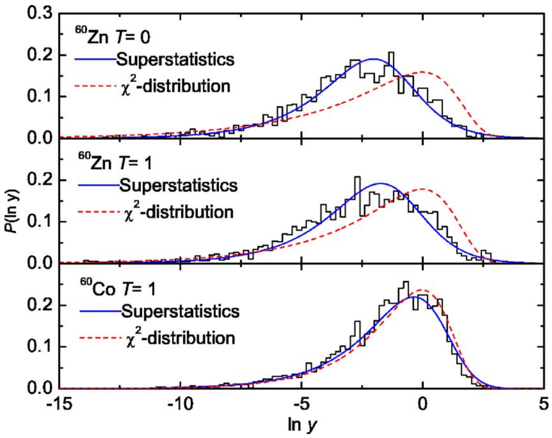

FIG. 2. (Color online) Nuclear shell-model M1 transition intensities in $A=60$, calculated by Hamoudi *et al.* [26], (histograms) compared with the superstatistical distribution (14) with parameters $\alpha=1.499, 0.064$, and 0.030, respectively (solid curves) and the χ^2 distribution (17) with parameters $\nu=1, 0.64$, and 0.34, respectively (dashed curves).

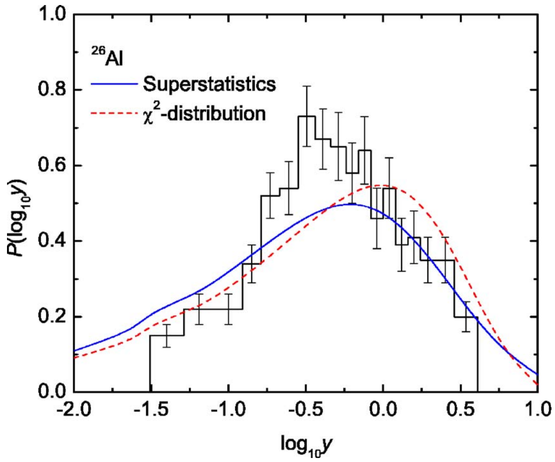

FIG. 3. (Color online) The distribution of experimental reduced transition probabilities in ^{26}Al from Ref. [24] (histogram) compared with the superstatistical distribution (14) with parameter $\alpha=1.24$ (solid curve) and the χ^2 distribution (17) with parameters $\nu=1.04$ (dashed curve).

the GOE statistics for the $T_z=0$ nuclei, where the matrix elements are purely isoscalar and relatively weak [33].

We analyzed the reduced M1 transition intensities for both the $T_z=1$ ^{60}Co nuclei and $T_z=0$ ^{60}Zn calculated by Hamoudi *et al.* [26] nuclei using the superstatistical transition intensity distribution in Eq. (14). These authors sampled a large number of matrix elements for each transition operator, which is equal to $56^2=3136$ and $66^2=4356$. Figure 2 compares the results of calculations using Eq. (14) with the numerical results of Hamoudi *et al.* [26] as well as the "best-fit" χ^2 distribution deduced by these authors. The figure clearly shows the advantage of the superstatistical distribution proposed here over the χ^2 distribution, at least for this numerical experiment.

The second work that we consider here is that of Adams, Mitchell, and Shriner [28]. They collected approximately 1500 experimental reduced electromagnetic transition strengths between the excited states of the nucleus ^{26}Al. Their data involve levels with isospin $T=0$ and $T=1$ between the ground state and the excitation energy of 8.067 MeV. Figure 3 compares these experimental data with results of calculations using Eq. (14) with $\alpha=1.24$ as well as the "best-fit" χ^2 distribution with a parameter ν slightly greater than 1. The figure again shows the advantage of the superstatistical distribution over the χ^2 distribution, although the agreement with the data is not as good as in the cases shown in Fig. 2. The experimental histogram is mostly higher than the theoretical curves especially in the peak region, although the data was normalized to 0.83 in order to approximately take care of the upper and lower detection thresholds. The percentage of undetected transitions may have been underestimated because its estimation was based on the Porter-Thomas distribution.

V. CONCLUSION

The eigenstates of a chaotic system are extended and cover the whole domain of classically permitted motion randomly, but uniformly. They overlap substantially, as manifested by level repulsion. There are no preferred eigenstates; the states are statistically equivalent. As a result, the matrix elements of transition operators in any basis are independent and have a Gaussian distribution, which leads to the Porter-Thomas distribution for reduced transition intensities. Coming out of the chaotic phase, the extended eigenstates become less and less homogeneous in space. Different eigenstates become localized in different places and the matrix elements that couple different pairs are no more statistically equal. The matrix elements will no longer have the same variance; one has to allow each of them to have its own variance. But this will dramatically increase the number of parameters of the theory. The proposed superstatistical approach solves this problem by treating all of the matrix elements as having a common variance, not fixed but fluctuating. One then expresses the probability density of transition intensities as an average of Porter-Thomas distributions with different mean intensities. The principle of maximum entropy is used to estimate the inverse-mean-intensity distribution. The resulting transition-intensity distribution is found to agree with the results of shell model calculation as well as with experimental data better that the χ^2 distribution, which is often used for this purpose.

[1] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, New York, 1991).
[2] F. Haake, *Quantum Signatures of Chaos* (Springer, Heidelberg, 1991).
[3] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[4] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960).
[5] M. S. Hussein and M. P. Pato, Phys. Rev. Lett. **70**, 1089 (1993); Phys. Rev. C **47**, 2401 (1993).
[6] M. S. Hussein and M. P. Pato, Phys. Rev. Lett. **80**, 1003 (1998).
[7] *AIM Conference Proceedings*, Vol. 777, p. 171 (1996).
[8] G. Casati, L. Molinari, and F. Izrailev, Phys. Rev. Lett. **64**, 1851 (1990).
[9] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67**, 2405 (1991).
[10] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[11] C. Tsallis, Lect. Notes Phys. **560**, 3 (2001).
[12] J. Evans and F. Michael, cond-mat/0207472 (unpublished); cond-mat/0208151 (unpublished).
[13] F. Toscano, R. O. Vallejos, and C. Tsallis, Phys. Rev. E **69**, 066131 (2004).
[14] A. C. Bertuola, O. Bohigas, and M. P. Pato, Phys. Rev. E **70**, 065102(R) (2004).
[15] F. D. Nobre and A. M. C. Souza, Physica A **339**, 354 (2004).
[16] A. Y. Abul-Magd, Phys. Lett. A **333**, 16 (2004).
[17] A. Y. Abul-Magd, Phys. Rev. E **71**, 066207 (2005).
[18] C. Beck and E. G. D. Cohen, Physica A **322**, 267 (2003).
[19] A. Y. Abul-Magd, Physica A **361**, 41 (2006).
[20] A. Y. Abul-Magd, Phys. Rev. E **72**, 066114 (2005).
[21] F. Sattin, Phys. Rev. E **68**, 032102 (2003).
[22] C. E. Porter, *Statistical Theory of Spectra; Fluctuation* (Academic Press, New York, 1965).
[23] Y. Alhassid and A. Novoselsky, Phys. Rev. C **45**, 1677 (1992).
[24] A. A. Adams, G. E. Mitchell, W. E. Ormand, and J. F. Shriner, Jr., Phys. Lett. B **392**, 1 (1997).
[25] C. I. Barbosa, T. Guhr, and H. L. Harney, Phys. Rev. E **62**, 1936 (2000).
[26] A. Hamoudi, R. G. Nazmitdinov, E. Shahaliev, and Y. Alhassid, Phys. Rev. C **65**, 064311 (2002).
[27] K. A. Muttalib and J. R. Klauder, Phys. Rev. E **71**, 055101(R) (2005).
[28] A. A. Adams, G. E. Mitchell, and J. F. Shriner, Jr., Phys. Lett. B **422**, 13 (1998).
[29] J. F. Shriner, Jr., C. A. Grossmann, and G. E. Mitchell, Phys. Rev. C **62**, 054305 (2000).
[30] Y. M. Luke, *The Special Functions and Their Applications* (Academic, New York, 1969), Vol. 1.
[31] Wolfram Research's Mathematical Functions 2002, http://functions.wolfram.com
[32] Y. Alhassid and R. D. Levine, Phys. Rev. Lett. **57**, 2879 (1986).
[33] P. J. Brussard and P. W. M. Glaudemans, *Shell-Model Applications in Nuclear Spectroscopy* (North-Holland, Amsterdam, 1977).